8-17

Recall from lecture:

- vectors
  - $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ (coordinate representation)
  - $\vec{v} = |\vec{v}| \cdot \dfrac{\vec{v}}{|\vec{v}|}$

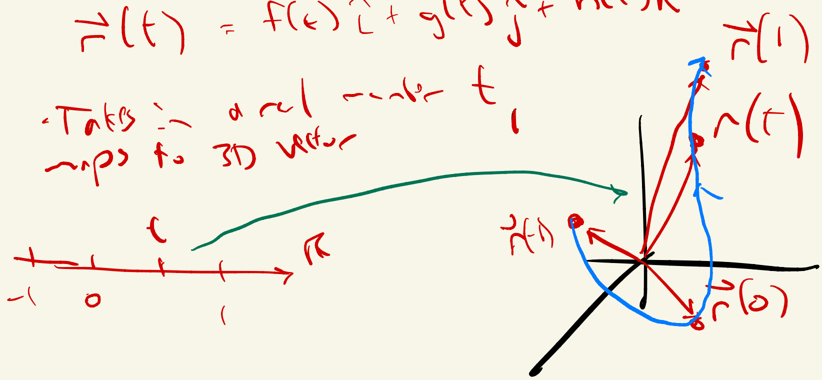
    magnitude (scalar measuring size)

    unit vector (direction)

- vector-valued functions
  - Function that eats e.g. scalars and spits out vectors
    - Ex: Parametrized curves

      $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

      - Takes in a real number t, maps to 3D vector



- Derivatives
  - unit tangent $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$ ← direction of velocity vector
  - unit normal $\vec{N} = \dfrac{d\vec{T}/dt}{|d\vec{T}/dt|}$ ← direction of change in tangent.

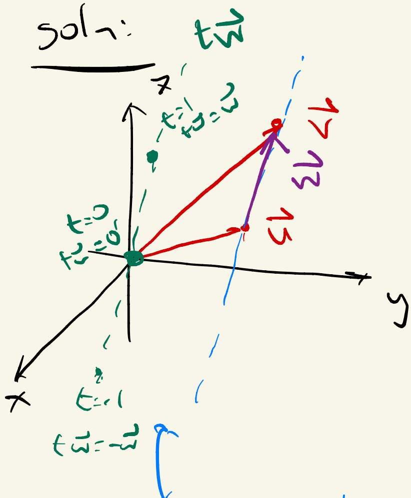
Parametrized Curves

① Find a parameterized curve between two points: (There are many parameterizations)

$\vec{u} = (u_1, u_2, u_3)$

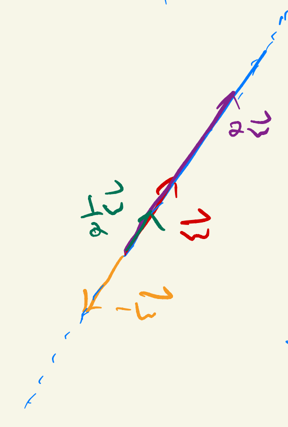
$\vec{v} = (v_1, v_2, v_3)$

soln:



line passing through $\vec{u}$ and $\vec{v}$.

Think about a line.



Take a vector parallel to the line.

Scalar multiplication
- resizes $\vec{w}$ and may walk in opposite direction (if scalar <0)

$\Rightarrow$ Traces out a line

$\{\text{scalar} \cdot \vec{w}\}$ = line parallel to $\vec{w}$

One vector that's parallel: $\vec{w}$

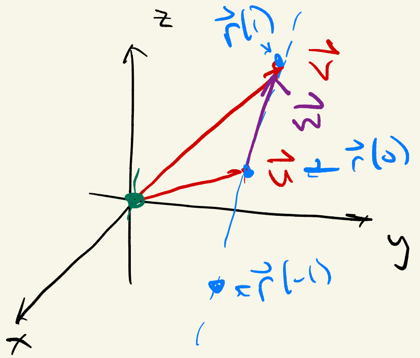
well $\vec{u} + \vec{w} = \vec{v}$

$\vec{w} = \vec{v} - \vec{u}$ is parallel to the line we want.

$\Rightarrow$ To get all scalar multiples of $\vec{w}$, we now use a parameter t. So, $t\vec{w}$ traces out this given line

$t\vec{w}$ is parallel to the line we want.

$$\Rightarrow \vec{r}(t) = \underset{\text{constant shift}}{\vec{u}} + \underbrace{t\vec{w}}_{\text{generates line}} = \vec{u} + t(\vec{v} - \vec{u})$$



$$\vec r(t) = \vec u + t(\vec v - \vec u)$$

Check a couple points:

$$\vec r(0) = \vec u + 0(\vec v - \vec u) = \vec u$$

$$\vec r(1) = \vec u + 1\cdot(\vec v - \vec u) = \vec u + \vec v - \vec u = \vec v$$

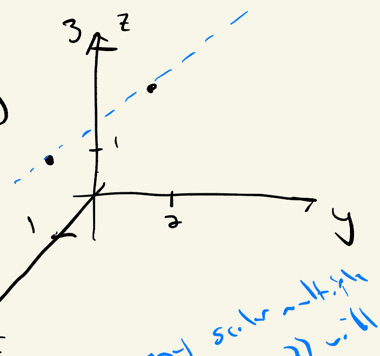
$$\vec r(-1) = \vec u + (-1)\cdot(\vec v - \vec u) = 2\vec u - \vec v$$

Ex:

Find a parametrization for the line passing through $(1,2,3)$ and $(-1,0,1)$
$\vec u$ $\qquad$ $\vec v$

Soln:

$$\vec r(t) = \vec u + t(\vec v - \vec u)$$
$$= (1,2,3) + t\big((-1,0,1) - (1,2,3)\big)$$
$$= (1,2,3) + t\big((-2,-2,-2)\big)$$
$$= (1,2,3) + t(-2,-2,-2)$$



works!

Possible alternatives

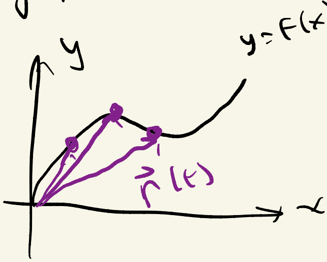
$$\vec r_1(t) = (1,2,3) + t(1,1,1)$$
$$\vec r_2(t) = (1,2,3) + t\left(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3}\right)$$
$$\vec r_3(t) = \vec v + t(\vec u - \vec v)$$
$$= (-1,0,1) + t\big((1,2,3) - (-1,0,1)\big)$$
$$= (-1,0,1) + t\big((2,2,2)\big)$$
$$= (-1,0,1) + t(2,2,2)$$

<u>any</u> scalar multiple of $(-2,-2,-2)$ will do the trick

$\vec u - \vec v = -(\vec v - \vec u)$ and so is on the same line

Another important class of parametrizations:

graphs of $y = f(x)$



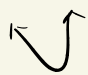
Each point is an $(x, y) = (x, f(x))$ coordinate.

Some trick for finding this parametrization always works: pick one variable, t, and

$$x(t) = t$$

$$y(t) = f(x(t)) = f(t)$$

$$\Rightarrow \boxed{\vec{r}(t) = t\,\hat{\imath} + f(t)\,\hat{\jmath}}$$



Ex: Curving parabola

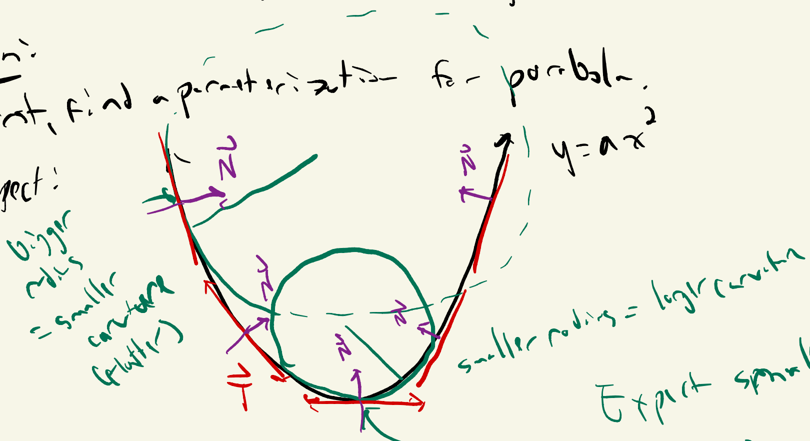
Find the x value for which the curve $y = ax^2$, $a > 0$ has the highest curvature κ.

Also, find unit normal and unit tangent for any point on the curve. What do you notice for large $\pm x$?

Soln:

First, find a parametrization for parabola.

Expect:



Expect smallest curvature at $x = 0$.

$\vec{T}$ - tangent (travel direction decides where $\vec{T}$ points)

$\vec{N}$ - normal

11:00

Trick:

$$\vec{r}(t) = t\,\hat{i} + f(t)\,\hat{j}$$
$$= t\,\hat{i} + at^2\,\hat{j}$$

$f(x) = ax^2$

$x(t) = t$

$y(t) = a\,(x(t))^2$

$= a\,t^2$

$-\infty < t < \infty$

Always works for graphs.

So let $\vec{r}(t) = \langle x(t), y(t) \rangle$
$= \langle t, at^2 \rangle$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 2at \rangle$$

$$\Rightarrow \vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 2at \rangle}{\sqrt{1+4a^2t^2}}$$

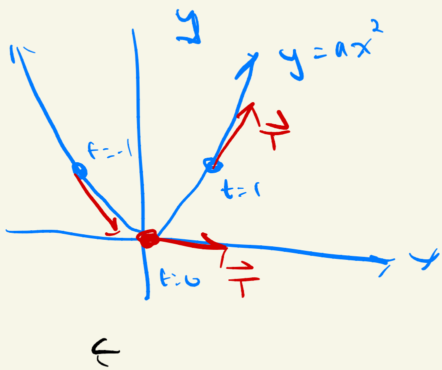
Note – ± direction of $\vec{T}$ depends on param

Does this make sense?

- At $x=0$, $t=0$
  $\Rightarrow \vec{T} = \langle 1, 0 \rangle$

- At $x=1$, $t=1$
  $\Rightarrow \vec{T} = \dfrac{\langle 1, 2a \rangle}{\sqrt{1+4a^2}}$

- At $x=-1$, $t=-1$
  $\Rightarrow \vec{T} = \dfrac{\langle 1, -2a \rangle}{\sqrt{1+4a^2}}$



what happens as $t \to \infty$ or $t \to -\infty$?

$$\lim_{t \to -\infty} \vec{T} = \lim_{t \to -\infty} \frac{\langle 1, 2at \rangle}{\sqrt{1+4a^2t^2}}$$

$$= \lim_{t \to -\infty} \left\langle \frac{1}{\sqrt{1+4a^2t^2}}, \frac{2at}{\sqrt{1+4a^2t^2}} \right\rangle$$

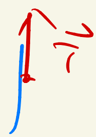
$$= \left\langle \lim_{t \to -\infty} \frac{1}{\sqrt{1+4a^2t^2}}, \lim_{t \to -\infty} \frac{2at}{\sqrt{1+4a^2t^2}} \right\rangle$$

$$= \langle 0, -1 \rangle$$

when t large,
$1+4a^2t^2 \approx 4a^2t^2$

$$\Rightarrow \frac{2at}{\sqrt{4a^2t^2}} = \frac{2at}{2a|t|}$$
$$= \frac{t}{|t|}$$

$t < 0$, $= -1$



If $t \to \infty$

$$\lim_{t \to \infty} \vec{T} = \lim_{t \to \infty} \frac{1}{\sqrt{1+4a^2t^2}} \langle 1, 2at \rangle$$

$$\text{"similarly"}_{t \to \infty} = \langle 0, 1 \rangle$$

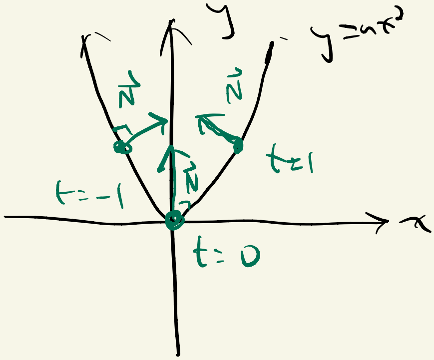
$$\frac{d\vec{T}}{dt} = \frac{1}{(1+4a^2t^2)^{3/2}} \langle -4a^2t, 2a \rangle$$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{\sqrt{16a^4t^2 + 4a^2}}{(1+4a^2t^2)^{3/2}}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{(1+4a^2t^2)^{3/2}}{\sqrt{16a^4t^2+4a^2}} \left\langle \frac{-4a^2t}{(1+4a^2t^2)^{3/2}}, \frac{2a}{(1+4a^2t^2)^{3/2}} \right\rangle$$

$$= \frac{1}{\sqrt{4a^2t^2+1}} \langle -2at, 1 \rangle$$

$$\vec{N} = \frac{1}{\sqrt{4a^2t^2+1}}\langle -2at, 1\rangle$$



check:

$x=t=0$

$$\vec{N} = \langle 0, 1\rangle$$

$x=t=-1$

$$\vec{N} = \frac{1}{\sqrt{4a^2+1}}\langle 2a, 1\rangle$$

$x=t=1$

$$\vec{N} = \frac{1}{\sqrt{4a^2+1}}\langle -2a, 1\rangle$$

limits:

$t \to -\infty$

$$\lim_{t\to-\infty} \vec{N} = \lim_{t\to-\infty}\left\langle \frac{-2at}{\sqrt{4a^2t^2+1}}, \frac{1}{\sqrt{4a^2t^2+1}}\right\rangle$$

large t
$4a^2t^2+1 \approx 4a^2t^2$

$$= \langle 1, 0\rangle$$

$t \to +\infty$

$$\lim_{t\to+\infty} \vec{N} = \langle -1, 0\rangle$$

Now for curvature:

$$\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{|\langle 1, 2at\rangle|}\left|\frac{1}{(1+4a^2t^2)^{3/2}}\langle -4a^2t, 2a\rangle\right|$$

$$= \frac{2a}{(1+4a^2t^2)^{3/2}}$$

Notice - only denominator depends on t!

and since $1+4a^2t^2 \geq 1$,

$$(1+4a^2t^2)^{3/2} \geq 1^{3/2} = 1$$

since $(\,)^{3/2}$ is increasing (as a function)

$\Rightarrow$ κ largest when $t=0$ and

$\kappa \to 0$ when $t \to \infty$

or $t \to -\infty$

$\Rightarrow t=0$ at $x=t=0$ $\boxed{\text{vertex}}$

$\Rightarrow y = ax^2 = 0$

κ largest at vertex, goes to 0 as $t \to \infty$

curvature $t \to -\infty$